

Electromagnetic response and pseudo-zero-mode Landau levels of bilayer graphene in a magnetic field

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The electromagnetic response of bilayer graphene in a magnetic field is studied in comparison with that of monolayer graphene. Both types of graphene turn out to be qualitatively quite similar in dielectric and screening characteristics, especially those deriving from vacuum fluctuations, but the effect is generally much more sizable for bilayers. The presence of the zero-(energy-)mode Landau levels is a feature specific to graphene. In bilayers, unlike in monolayers, the effect of the zero-mode levels becomes visible and even dominant in density response as an externally-controllable band gap develops. It is pointed out that the splitting of nearly-degenerate pseudo-zero-mode levels at each valley, specific to bilayer graphene, is controlled by an applied inplane electric field or by an injected current. In addition, a low-energy effective gauge theory of bilayer graphene is constructed.

PACS numbers: 73.43.-f, 71.10.Pm, 77.22.Ch

I. INTRODUCTION

Graphene, a monolayer of graphite, has recently been attracting great attention, both experimentally^{1,2,3} and theoretically,^{4,5,6,7,8} for its unusual electronic transport, characteristic of “relativistic” charge carriers behaving like massless Dirac fermions. Graphene is naturally of interest from the viewpoint of relativistic quantum field theory, and is a special laboratory to test the particle-hole picture⁹ of the quantum vacuum and, especially in a magnetic field, to study peculiar quantum phenomena^{10,11,12,13,14} tied to the chiral and parity anomalies. Actually, the half-integer quantum Hall (QH) effect and the presence of the zero-energy Landau levels observed^{1,2} in graphene are a manifestation of spectral asymmetry implied by the anomaly.

The bilayer (and multilayer) of graphene is as interesting and exotic^{15,16} as the monolayer. In bilayer graphene interlayer coupling modifies the intralayer relativistic spectra to yield a quasiparticle spectrum with a parabolic energy dispersion.¹⁵ The relativistic feature thereby disappears but the particle-hole structure still remains, leading to a “chiral” Schroedinger Hamiltonian, which has no analog in particle physics. Bilayer graphene, like monolayers, is intrinsically a gapless semiconductor but has a notable property that the energy gap between the conduction and valence bands is controllable^{17,18,19,20,21} by use of external gates or chemical doping.

Theoretical studies show that electronic transport^{4,5,6} and screening²² in graphene are substantially different from those in standard planar systems. The difference becomes even more prominent under a magnetic field.²³ For graphene the vacuum state is a dielectric medium with appreciable electric and magnetic susceptibilities over all range of wavelengths. Curiously the zero-energy Landau levels, though carrying normal Hall conductance e^2/h per level, scarcely contribute to the dielectric effect.

The purpose of this paper is to study the electromag-

netic response of bilayer graphene in a magnetic field at integer filling factor ν , in comparison with that of monolayer graphene. It turns out that both types of graphene are qualitatively quite similar in dielectric and screening characteristics, but the effect is much more sizable for bilayers (because of the difference in the basic cyclotron energy).

In bilayer graphene there arise two species of zero-energy levels (or zero modes), which, when an interlayer field is applied, move up or down (oppositely) at the two valleys. Remarkably, the dielectric effects due to the (pseudo) zero modes, unlike in monolayers, become visible and even grow steadily as the tunable band gap develops. It is pointed out that the splitting of nearly degenerate pseudo-zero-mode levels at each valley, specific to bilayer graphene, is controlled by an inplane electric field or by an injected current. In addition, we construct out of the response a low-energy effective gauge theory of bilayer graphene in a magnetic field and verify that the electric susceptibility of a QH system is generally expressed as a ratio of the Hall conductance to the Landau gap.

In Sec. II we briefly review the low-energy effective theory of bilayer graphene and study its Landau-level spectrum. In Sec. III we examine the electromagnetic response and screening properties of bilayer graphene. In Sec. IV we derive an effective gauge theory. In Sec. V we study the effect of an inplane field on the almost degenerate zero-mode levels at each valley and discuss its consequences. Section VI is devoted to a summary and discussion.

II. BILAYER GRAPHENE

The bilayer graphene consists of two coupled hexagonal lattices of carbon atoms, arranged in Bernal $A'B$ stacking, with inequivalent sites denoted as (A, B) in the bottom layer and (A', B') in the top layer. The elec-

tron fields in the bilayer are described by four-component spinors on the four sites and, as in the case of monolayer graphene, their low-energy spectrum is governed by the electron states near the two inequivalent Fermi points K and K' in the Brillouin zone. The intralayer coupling $\gamma_0 \equiv \gamma_{AB} \approx 2.9$ eV is related to the Fermi velocity in monolayer graphene, $v_0 = (\sqrt{3}/2) a_L \gamma_0 / \hbar \approx 10^6$ m/s, with the lattice constant $a_L = 0.246$ nm. The interlayer coupling $\gamma_1 \equiv \gamma_{A'B}$ and $\gamma_3 \equiv \gamma_{AB'}$ are one-order of magnitude weaker than the intralayer coupling γ_0 ; numerically,²⁴ $\gamma_1 \approx 0.30$ eV and $\gamma_3 \approx 0.10$ eV.

Actually the neighboring sites A' and B form interlayer dimers via γ_1 and get shifted to higher energy bands, and the low-energy sector is essentially described by two-component spinors residing on the A and B' sites. The effective Hamiltonian is written as¹⁵

$$\begin{aligned} H &= \int d^2 \mathbf{x} \left[\psi^\dagger (\mathcal{H}_+ - eA_0) \psi + \chi^\dagger (\mathcal{H}_- - eA_0) \chi \right], \\ \mathcal{H}_\xi &= \mathcal{H}_0 + \mathcal{H}_{\text{as}}, \\ \mathcal{H}_0 &= \xi v_3 \begin{pmatrix} & \Pi \\ \Pi^\dagger & \end{pmatrix} - \frac{1}{2m^*} \begin{pmatrix} & (\Pi^\dagger)^2 \\ \Pi^2 & \end{pmatrix}, \\ \mathcal{H}_{\text{as}} &= \frac{\xi U}{2} \left[\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} - \frac{1}{\gamma_1 m^*} \begin{pmatrix} \Pi^\dagger \Pi & \\ & -\Pi \Pi^\dagger \end{pmatrix} \right], \end{aligned} \quad (2.1)$$

where coupling to electromagnetic potentials (A_i, A_0) is introduced through $\Pi = \Pi_x - i\Pi_y$, $\Pi^\dagger = \Pi_x + i\Pi_y$, and $\Pi_i = -i\partial_i + eA_i$. Here $\xi = 1$ refers to the K valley with $\mathcal{H}_+ = \mathcal{H}_{\xi=1}$ and $\psi = (\psi_A, \psi_{B'})^t$ while $\xi = -1$ refers to the K' valley with $\mathcal{H}_- = \mathcal{H}_{\xi=-1}$ and $\chi = (\chi_{B'}, \chi_A)^t$. We suppress the electron spin, which is taken care of simply via spin degeneracy $g_s = 2$.

In \mathcal{H}_0 the first term with a linear dispersion represents direct interlayer hopping via γ_3 , with characteristic velocity $v_3 = (\sqrt{3}/2) a_L \gamma_3 / \hbar \sim v_0/30$, while the second term with a quadratic dispersion represents $A \leftrightarrow B'$ hopping via the dimer state, giving $\hbar^2/(2m^*) = v_0^2/\gamma_1$.

The \mathcal{H}_{as} takes into account a possible asymmetry between the two layers, leading to a gap U between the conduction and valence bands. An important feature of bilayer graphene is that such a gap is controllable^{17,18,19} by use of external gates, $U \approx e\Delta A_0$ with an interlayer voltage $\Delta A_0 = A_0^{\text{top}} - A_0^{\text{bottom}}$. The second term in \mathcal{H}_{as} is a layer asymmetry associated with the depleted charge on the AB' dimer sites; we call it a kinetic asymmetry.

Let us place graphene in a strong uniform magnetic field $B > 0$ normal to the sample plane; to this end we set $A_i(x) \rightarrow \mathbf{A}^B = B(-y, 0)$. It is convenient to rescale $\Pi = \sqrt{2eB}a = (\sqrt{2}/\ell)a$ and $\Pi^\dagger = (\sqrt{2}/\ell)a^\dagger$ with the magnetic length $\ell = 1/\sqrt{eB}$ so that $[a, a^\dagger] = 1$. The kinetic terms thereby acquire the scale $v_3 \rightarrow v_3 \sqrt{2}/\ell \equiv \omega_3$ and $1/(2m^*) \rightarrow eB/m^* \equiv \omega_c$; numerically, $\omega_c \approx 3.9 \times B[\text{T}]$ meV and $\omega_3 \approx 1.2 \times \sqrt{B[\text{T}]} \text{ meV}$, where $B[\text{T}]$ stands for the magnetic field in Tesla. The associated Landau-level spectra scale like $\omega_3 \sqrt{n}$ and $\omega_c \sqrt{n(n-1)}$ with the level index $n = 0, 1, \dots$, with ratio $\sim 0.3/(\sqrt{n-1}\sqrt{B[\text{T}]})$. Therefore the ω_3 term is practically negligible, compared

with the ω_c term, for higher Landau levels $n \geq 2$ under a strong magnetic field. With this in mind we cast the Hamiltonian \mathcal{H}_ξ in the form

$$\mathcal{H}_\xi = \omega_c \begin{pmatrix} \mu(1 - z a^\dagger a) & \lambda a - (a^\dagger)^2 \\ \lambda a^\dagger - a^2 & -\mu(1 - z a a^\dagger) \end{pmatrix}, \quad (2.2)$$

with $\lambda = \xi \omega_3 / \omega_c \approx \pm 0.3 / \sqrt{B[\text{T}]}$. Here $\mu = \xi U / (2\omega_c)$ stands for half of the band gap in units of ω_c , and it seems feasible¹⁸ to achieve an interlayer-voltage change of magnitude $\mu \sim O(1)$. Note that the kinetic asymmetry $\sim \mu z a^\dagger a$ is very weak, with $z = 2\omega_c / \gamma_1 \approx 0.026 \times B[\text{T}] \ll 1$.

Let us for the moment set a tiny parameter $z \rightarrow 0$. It is, then, generally seen from the structure of \mathcal{H}_ξ that its eigenmodes are the same as those of $\mathcal{H}_\xi|_{\mu \rightarrow 0}$ and that the spectrum of \mathcal{H}_ξ is symmetric about $\epsilon = 0$, except for a possible $\epsilon = \omega_c \mu$ spectrum or the zero-energy spectrum of $\mathcal{H}_\xi|_{\mu \rightarrow 0}$. We shall call the $\epsilon = \pm \omega_c |\mu|$ eigenmodes of \mathcal{H}_ξ pseudo-zero-modes, or simply "zero" modes.

Such spectra are slightly modulated by the tiny $O(\mu z)$ kinetic asymmetry. For $\lambda = 0$ it is possible to write down the eigenmodes explicitly. The "nonzero" modes of \mathcal{H}_ξ are Landau levels of energy

$$\epsilon_n = s_n \omega_c \sqrt{|n|(|n| - 1) + \hat{\mu}_n^2} - \frac{1}{2} \omega_c \mu z \quad (2.3)$$

labeled by integers $n = \pm 2, \pm 3, \dots$, and p_x (or $y_0 \equiv \ell^2 p_x$); $\hat{\mu}_n = \mu \{1 - (|n| - \frac{1}{2})z\}$. Here $s_n \equiv \text{sgn}\{n\} = \pm 1$ specifies the sign of the energy ϵ_n . The associated eigenmodes are written as

$$\psi_{ny_0}(\mathbf{x}) = \frac{1}{\sqrt{2}} \left(c_n^+ \phi_{|n|}(\mathbf{x}), -s_n c_n^- \phi_{|n|-2}(\mathbf{x}) \right)^t, \quad (2.4)$$

where $c_n^\pm = \sqrt{1 \pm \hat{\mu}_n / \epsilon_n'}$ and $\epsilon_n' = s_n \sqrt{|n|(|n| - 1) + \hat{\mu}_n^2}$; $\phi_n(\mathbf{x}) = \phi_n(y - y_0) (e^{ixy_0/\ell^2} / \sqrt{2\pi\ell^2})$ are the eigenfunctions for the Landau levels (n, y_0) of the usual Hall electron.

In bilayer graphene there arise two nearly-degenerate zero-mode levels per valley and spin, with spectrum

$$\epsilon_0 = \omega_c \mu, \quad \epsilon_1 = \omega_c \mu (1 - z), \quad (2.5)$$

and eigenfunctions $\psi_{0y_0} = (\phi_0(\mathbf{x}), 0)^t$ and $\psi_{1y_0} = (\phi_1(\mathbf{x}), 0)^t$. We take, without loss of generality, $U/\omega_c = 2\xi\mu > 0$ and label the Landau levels associated with the $\xi = 1$ valley (i.e., the ψ sector with $\mu > 0$) by $n = 0_+, 1, \pm 2, \dots$ and those associated with the $\xi = -1$ valley (the χ sector with $\mu < 0$) by $n = 0_-, -1, \pm 2, \dots$. Note that the zero-mode spectrum is ordered according to $\epsilon_{0_-} < \epsilon_{-1} < 0 < \epsilon_1 < \epsilon_{0_+}$; see Fig. 1. An interlayer voltage $U \propto 2\mu$ thus works to shift the zero modes oppositely at the two valleys, opening a gap U , while the nonzero-mode levels get shifted only slightly and remain symmetric, apart from the $O(\mu z)$ asymmetry.

The zero-mode levels at each valley are degenerate for $z \rightarrow 0$ and the degeneracy remains even in the presence

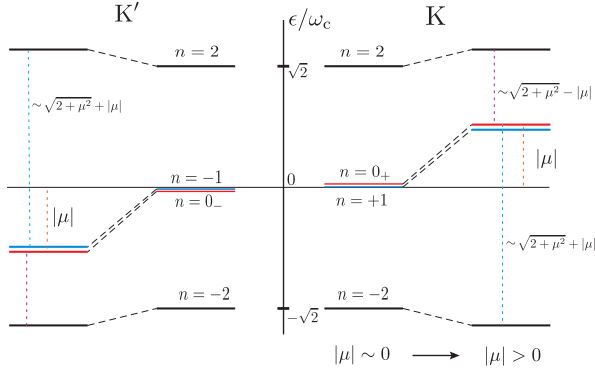


FIG. 1: (Color online) Landau-level spectrum. The zero-mode levels move almost linearly with the interlayer voltage $\propto \mu$ and oppositely at the two valleys while other levels are shifted only slightly.

of the linear kinetic term. This is because the zero modes persist even for $\lambda \neq 0$; indeed, $D\phi = 0$ with $D = \lambda a^\dagger - a^2$ has two independent solutions of power series in λ ,

$$\begin{aligned}\phi_{0'}(\mathbf{x}) &= (1/N_0) \sum_{n=0}^{\infty} \alpha_n \phi_{3n}(\mathbf{x}), \\ \phi_{1'}(\mathbf{x}) &= (1/N_1) \sum_{n=0}^{\infty} \zeta_n \phi_{3n+1}(\mathbf{x}), \\ \alpha_n &= \lambda^n \sqrt{\frac{\Gamma(2/3) \Gamma(n+1/3)}{3^n n! \Gamma(n+2/3) \Gamma(1/3)}},\end{aligned}\quad (2.6)$$

with normalization factors $N_0^2 = \sum_n \alpha_n^2$ and $N_1^2 = \sum_n \zeta_n^2$; ζ_n is given by α_n with replacement $2/3 \rightarrow 4/3$ and $1/3 \rightarrow 2/3$ in the argument of the gamma functions. These $n = 0'_+, 1'$ solutions have an infinite radius of convergence in λ . This verifies that the index of the Dirac Hamiltonian, $\text{Index}[\mathcal{H}_\xi|_{\mu \rightarrow 0}] = \dim \text{Ker} D - \dim \text{Ker} D^\dagger$, is two, apart from the Landau-level degeneracy,

$$\text{Index}[\mathcal{H}_\xi|_{\mu \rightarrow 0}] = 2 \times \frac{1}{2\pi\ell^2} \int d^2\mathbf{x} = \int d^2\mathbf{x} \frac{eB}{\pi}. \quad (2.7)$$

The zero-mode spectrum, corrected by the $O(\mu z)$ asymmetry, eventually reads

$$\begin{aligned}\epsilon_{0'} &= \omega_c \mu \{1 - z b_0(\lambda)\}, \\ \epsilon_{1'} &= \omega_c \mu \{1 - z b_1(\lambda)\},\end{aligned}\quad (2.8)$$

with $b_0(\lambda) = \sum_n 3n\alpha_n^2 / \sum_n \alpha_n^2$ and $b_1(\lambda) = \sum_n (3n+1)\zeta_n^2 / \sum_n \zeta_n^2$, or

$$\begin{aligned}b_0(\lambda) &= (1/2)\lambda^2 + 0.05\lambda^4 + \dots, \\ b_1(\lambda) &= 1 + (1/2)\lambda^2 - 0.036\lambda^4 + \dots.\end{aligned}\quad (2.9)$$

There is a common level shift of $O(z\lambda^2)$ while the tiny level splitting $\sim \omega_c \mu z \ll \omega_c$ is practically unchanged (for $\lambda \sim 0.3$). It is thus generally difficult to resolve the

almost degenerate zero-mode levels; in a sense, it is the nonzero index (2.7) of the Hamiltonian \mathcal{H}_ξ that underlies this stability in the zero-mode degeneracy.

In connection with the index, it would be worth remarking that $D^\dagger \phi = 0$ has no solution; a solution of infinite power series in $1/\lambda$, starting with $\phi_0(\mathbf{x})$, fails to converge for finite λ . Actually the $\lambda \rightarrow \infty$ limit corresponds to the case of monolayer graphene with linear dispersion. The index thus jumps from "2" to "-1" as one passes from finite λ to infinite λ .

The linear kinetic term affects the spectrum ϵ_n of the $|n| \geq 2$ levels only to $O(\lambda^2)$, which is still negligible in a strong magnetic field. We shall therefore set $\lambda \rightarrow 0$ in most of our analysis below, except in Sec. V where we discuss possible resolution of the zero-mode levels.

For actual calculations it is useful to make the Landau-level structure explicit via the expansion^{25,26} $\psi(\mathbf{x}, t) = \sum_{n, y_0} \langle \mathbf{x} | n, y_0 \rangle \psi_n(y_0, t)$. (From now on, we shall only display the ψ sector since the χ sector is obtained by reversing the signs of μ and λ .) The Hamiltonian H thereby is rewritten as

$$H = \int dy_0 \sum_{n=-\infty}^{\infty} \psi_n^\dagger \epsilon_n \psi_n, \quad (2.10)$$

and the charge density $\rho_{-\mathbf{p}}(t) = \int d^2\mathbf{x} e^{i\mathbf{p}\cdot\mathbf{x}} \psi^\dagger \psi$ as

$$\rho_{-\mathbf{p}} = e^{-\ell^2 \mathbf{p}^2 / 4} \sum_{k, n=-\infty}^{\infty} g_{kn}(\mathbf{p}) \int dy_0 \psi_k^\dagger e^{i\mathbf{p}\cdot\mathbf{r}} \psi_n, \quad (2.11)$$

with the coefficient matrix

$$\begin{aligned}g_{kn}(\mathbf{p}) &= \frac{1}{2} \left[c_k^+ c_n^+ f_{|k|, |n|}(\mathbf{p}) \right. \\ &\quad \left. + s_k s_n c_k^- c_n^- f_{|k|-2, |n|-2}(\mathbf{p}) \right];\end{aligned}\quad (2.12)$$

$\mathbf{r} = (r_1, r_2) = (i\ell^2 \partial / \partial y_0, y_0)$ stands for the center coordinate with uncertainty $[r_1, r_2] = i\ell^2$. Here

$$f_{kn}(\mathbf{p}) = \sqrt{\frac{n!}{k!}} \left(\frac{i\ell p}{\sqrt{2}} \right)^{k-n} L_n^{(k-n)} \left(\frac{1}{2} \ell^2 \mathbf{p}^2 \right) \quad (2.13)$$

for $k \geq n$, and $f_{nk}(\mathbf{p}) = [f_{kn}(-\mathbf{p})]^\dagger$; $p = p_y + ip_x$; actually $f_{kn}(\mathbf{p})$ are the coefficient functions for the charge density of the ordinary Hall electrons.

In a similar fashion one can derive the expression for the current operator $\mathbf{j} \sim \delta H / \delta \mathbf{A}$. We omit it here, and simply remark that the current has a component coming from the $O(\mu z)$ asymmetry as well.

III. ELECTROMAGNETIC RESPONSE

In this section we study the electromagnetic response of bilayer graphene. Let us first consider the polarization function $P(\mathbf{p}, \omega) (\sim -i \langle \rho \rho \rangle)$ in Fourier space,

$$P(\mathbf{p}, \omega) = - \sum_{k, n} \left\{ \frac{1}{\epsilon_{kn} - \omega} + \frac{1}{\epsilon_{kn} + \omega} \right\} \sigma_n^k(\mathbf{p}),$$

$$\sigma_n^k(\mathbf{p}) = \frac{1}{2\pi\ell^2} e^{-\ell^2 \mathbf{p}^2/2} |g_{kn}(\mathbf{p})|^2, \quad (3.1)$$

where $\epsilon_{kn} = \epsilon_k - \epsilon_n$; the sum is taken over occupied levels $\{n\}$ and unoccupied levels $\{k\}$. In what follows we focus on the real part of $P(\mathbf{p}, \omega)$ in the static limit $\omega \rightarrow 0$, and denote the components coming from the virtual ($n \rightarrow k \rightarrow n$) transitions as

$$P_n^k(\mathbf{p}) = -\frac{1}{\pi\ell^2} \frac{1}{\epsilon_{kn}} e^{-\ell^2 \mathbf{p}^2/2} |g_{kn}(\mathbf{p})|^2. \quad (3.2)$$

Actually $|g_{kn}(\mathbf{p})|^2 = g_{nk}(-\mathbf{p}) g_{kn}(\mathbf{p})$ are functions of $\ell^2 \mathbf{p}^2$ and are thus symmetric in (k, n) , which implies the relation $P_n^k = -P_k^n$.

For conventional QH systems the polarization function vanishes for the vacuum since the charge operator trivially annihilates the vacuum $\rho|\nu=0\rangle=0$. For graphene, in contrast, $\rho|\nu=0\rangle \neq 0$ because of pair creation from the Dirac sea, and even the vacuum acquires nonzero polarization

$$P(\mathbf{p}, 0)|_{\nu=0} = \sum_{k=0}^N \sum_{n=2}^N P_{-n}^k(\mathbf{p}). \quad (3.3)$$

Some care is needed in carrying out sums over an infinite number of Landau levels, which are potentially singular. For regularization we here choose, as before,²³ to truncate the spectrum to a finite interval $-N \leq n, k \leq N$ and let $N \rightarrow \infty$ in the very end. Regularization not only keeps calculations under control but also refines them: Actually, instead of Eq. (3.3), one may first consider, for a given level n , the polarization "per level" $P_n = \sum_{k=-N}^N P_n^k$ by summing over all levels k but n , and then obtain $P(\mathbf{p}, 0) = \sum_n P_n$ by summing over filled levels n . This gives the same result as Eq. (3.3) owing to the antisymmetry $P_n^k = -P_k^n$.

Equation (3.3) is an expression for the ψ sector ($P \rightarrow P^\psi$). For the charge operator $\rho^\chi = \chi^\dagger \chi$ in the χ sector one may replace, in Eq. (2.11), $g_{kn}(\mathbf{p})$ by

$$g_{kn}^\chi(\mathbf{p}) = g_{-k, -n}(\mathbf{p}), \quad (3.4)$$

i.e., with the sign of μ reversed. One can define $(P^\chi)_n^k(\mathbf{p})$ accordingly and write the vacuum polarization function as $P^\chi(\mathbf{p}, 0)|_{\nu=0} = \sum_{k=2}^N \sum_{n=0}^N (P^\chi)_n^k(\mathbf{p})$. Note that these components enjoy the following property

$$(P^\psi)_n^k = -(P^\psi)_k^{-n} = (P^\chi)_n^k. \quad (3.5)$$

This implies, in particular, that the ψ and χ sectors contribute equally to the vacuum polarization $P^\psi(\mathbf{p}, 0)|_{\nu=0} = P^\chi(\mathbf{p}, 0)|_{\nu=0}$, and also that $P(\mathbf{p}, 0) = P^\psi(\mathbf{p}) + P^\chi(\mathbf{p})$ is the same for the charge-conjugate states with $\nu = \pm$ integer. (In view of this, we shall focus on the case $\nu \geq 0$ from now on.)

In calculating the density response we suppose that the almost degenerate zero-mode levels at each valley remain practically inseparable and treat them as both occupied or empty; accordingly we set $z \rightarrow 0$ below. Let us first

look into the leading long-wavelength part $\sim O(\mathbf{p}^2)$ of $P(\mathbf{p}, 0)$, to be denoted as $P^{(2)}(\mathbf{p}, 0)$, which is related to the electric susceptibility $\alpha_e = -(e^2/\mathbf{p}^2) P^{(2)}(\mathbf{p}, 0)$. A look into the matrix elements in Eq. (2.12) shows that $P_n^{(2)}(\mathbf{p}, 0)$ derives only from virtual transitions to the adjacent levels ($n \rightarrow n \pm 1$) and the related ones across the Dirac sea ($n \rightarrow -n \pm 1$). A direct calculation then yields

$$P_n^{(2)} = -\frac{\mathbf{p}^2}{2\pi\omega_c} \left(\beta_n - \frac{\mu}{|n|(|n|-1)} \right), \quad (3.6)$$

$$\beta_n = s_n \frac{|n| - 1/2}{\sqrt{|n|(|n|-1) + \mu^2}}, \quad (3.7)$$

for $2 \leq |n| \leq N-1$, and

$$P_1^{(2)} + P_{0+}^{(2)} = -(\mathbf{p}^2/2\pi\omega_c) 2|\mu|. \quad (3.8)$$

The bottom of the Dirac sea yields $P_{-N}^{(2)} = (P^{(2)})_{-N}^{N-1} + (P^{(2)})_{-N}^{-(N-1)} \propto N-1 + O(1/N)$, which properly makes $P^{(2)}|_{\nu=0} = \sum_{n=2}^N P_{-n}^{(2)}$ finite. This leads to the vacuum electric susceptibility (per valley and spin)

$$\alpha_e^{\text{vac}} = \frac{e^2}{2\pi\omega_c} F(\mu), \quad (3.9)$$

$$F(\mu) = -\sum_{n=2}^{N-1} \beta_n - |\mu| + N-1, \\ \approx 0.87715 - |\mu| + O(\mu^2). \quad (3.10)$$

An alternative expression for this α_e^{vac} is obtained by summing up the virtual ($-n \rightarrow n \pm 1$) processes, or $\sum_{n=2}^N P_{-n}^{n-1} + \sum_{n=2}^{N-1} P_{-n}^{n+1}$, which yields

$$F(\mu) = \sum_{n=1}^{N-1} \frac{1}{4\epsilon'_n \epsilon'_{n+1}} \left[2(\epsilon'_{n+1} - \epsilon'_n) - (\epsilon'_{n+1} - \epsilon'_n)^3 \right] \\ = \sum_{n=1}^{N-1} \frac{(\sqrt{n+1} - \sqrt{n-1})^3}{4\sqrt{n}} - |\mu| + O(\mu^2), \quad (3.11)$$

where $\epsilon'_n = \sqrt{n(n-1) + \mu^2}$.

Equations (3.6) and (3.8) tell us that the susceptibility carried by a positive-energy level of the same n is slightly different by $O(\mu)$ terms at the two valleys. Such $O(\mu)$ corrections are visible only for the zero-mode levels since the two valleys are practically indistinguishable for $n \geq 2$ (up to $O(\mu z)$ splitting). Let us suppose that the electrons fill up an integral number ν of Landau levels, with uniform density $\langle \rho \rangle \equiv \bar{\rho} = \nu/(2\pi\ell^2)$. We write $\nu = \sum_n \nu_n$ in terms of the filling factors ν_n of the n th level [$0 \leq \nu_n \leq 4$ for $n \geq 2$ and $0 \leq \nu_{\{1,0\}} \leq 4$; $\nu_{\{1,0\}} \equiv \nu_1 + \nu_{0+}$], with both valley and spin taken into account. The electric susceptibility at integer filling ν is then written as

$$\alpha_e = \frac{e^2}{2\pi\omega_c} \left\{ 4F(\mu) + \nu_{\{1,0\}} |\mu| + \sum_{n \geq 2} \nu_n \beta_n \right\}. \quad (3.12)$$

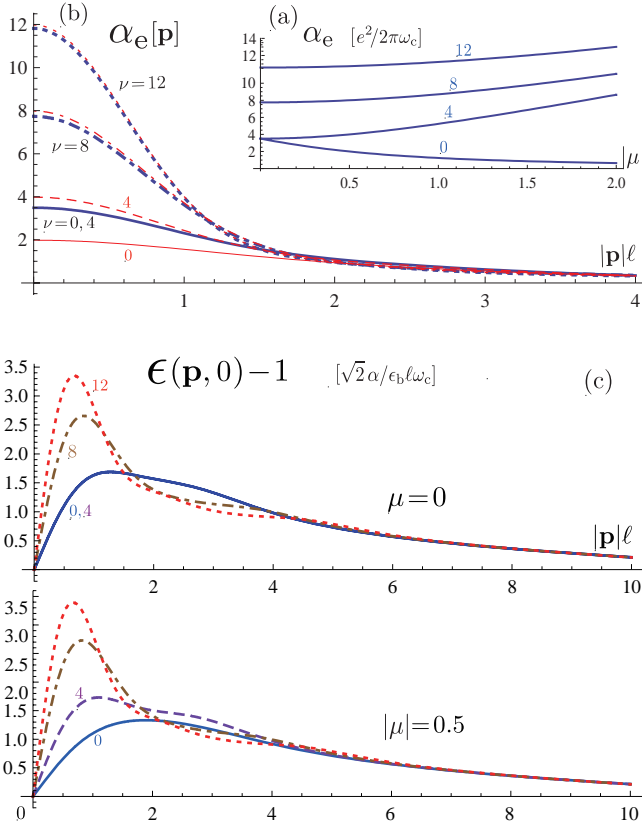


FIG. 2: (Color online) (a) Electric susceptibility α_e decreases with the band gap $2|\mu|\omega_c$ for $\nu = 0$ and rises for $\nu = 4, 8$ and 12 . (b) Static susceptibility function $\alpha_e[\mathbf{p}]$, in units of $e^2/(2\pi\omega_c)$, at $\nu = 0, 4, 8, 12$ for zero band gap $\propto \mu = 0$ (thick curves) and for a finite gap $|\mu| = 0.5$ (thin red curves). (c) Static dielectric function $\epsilon(\mathbf{p}, 0) - 1$, in units of $\sqrt{2}\alpha/(\epsilon_b\omega_c\ell)$, at $\nu = 0 \sim 12$ for a band gap $\propto \mu = 0$ and $|\mu| = 0.5$.

(This α_e is even in ν , and applies to the case of holes with $\nu \leq 0$ equally well.)

It is amusing to note here that some expressions greatly simplify for a special value $\mu = 1/2$: The spectrum is equally spaced, $\epsilon_n = s_n\omega_c(|n| - 1/2)$ for $|n| \geq 2$ and $\epsilon_1 = \epsilon_{0+} = \omega_c/2$, apart from the $O(\mu z)$ splitting. For the susceptibility one finds $\beta_n = 1$ for $n \geq 2$, and $F(1/2) = 1/2$. This yields

$$\alpha_e|_{\mu=1/2} = (2\delta_{\nu 0} + \nu) e^2/(2\pi\omega_c), \quad (3.13)$$

which rises linearly with $\nu = 4, 8, 12, \dots$.

In Fig. 2 (a) we plot α_e as a function of the band gap $2|\mu|\omega_c$ for $\nu = 0, 4, 8$ and 12 . The vacuum susceptibility $\alpha_e^{\text{vac}} = \alpha_e|_{\nu=0}$ is almost comparable to the contribution of the filled $n = 2$ level for $\mu \sim 0$. The α_e^{vac} decreases gradually as the band gap $\propto \mu$ develops; this is clear intuitively. In contrast, when the zero-mode levels get filled, i.e., at $\nu = 4$, α_e starts to grow almost linearly with μ , and such characteristic behavior persists for higher ν as well. This is somewhat unexpected but is easy to understand: The zero-mode levels move linearly with the band

gap $\propto |\mu|$ while the $n \geq 2$ levels are shifted only slowly, as seen from Fig. 1. The growth of α_e is therefore due to a decrease in activation gap from the zero-mode levels. This implies that the dielectric effect due to the zero modes, though negligible for $|\mu| \ll 1$, becomes dominant for large gaps $|\mu| \sim 1$.

This is also the case with the full expression for the polarization $P(\mathbf{p}, 0)$, which one can evaluate numerically using Eqs. (2.12) and (3.2). In Fig. 2 (b) we plot the susceptibility function $\alpha_e[\mathbf{p}] = -(e^2/\mathbf{p}^2) P(\mathbf{p}, 0)$ at $\nu = 0, 4, 8, 12$ for zero band gap and for a finite gap. There, as in the monolayer case, the zero-mode levels scarcely contribute to $\alpha_e[\mathbf{p}]$ for zero gap, but one sees clearly that, as the band gap $\propto \mu$ develops, they make $\alpha_e[\mathbf{p}]$ distinct between the vacuum and the $\nu = 4$ state.

The susceptibility is also related to screening properties of graphene. Let us now turn on the Coulomb interaction $v = \alpha/(\epsilon_b|\mathbf{x}|)$ or $v_{\mathbf{p}} = 2\pi\alpha/(\epsilon_b|\mathbf{p}|)$ with $\alpha = e^2/(4\pi\epsilon_0) \approx 1/137$ and the substrate dielectric constant ϵ_b , and study its effects in the random-phase approximation (RPA). The RPA dielectric function is written as²⁷

$$\epsilon(\mathbf{p}, \omega) = 1 - v_{\mathbf{p}} P(\mathbf{p}, \omega). \quad (3.14)$$

Figure 2 (c) shows the static function $\epsilon(\mathbf{p}, 0) - 1$ for $\nu = 0, 4, 8, 12$, plotted in units of $\sqrt{2}\alpha/(\epsilon_b\omega_c\ell)$. Note first that there is no screening at long distances, $\epsilon(\mathbf{p}, 0) \rightarrow 1$ for $\mathbf{p} \rightarrow 0$, as is typical of two-dimensional systems. As wave vector $|\mathbf{p}|$ is increased, $\epsilon(\mathbf{p}, 0)$ grows rapidly, becomes sizable for $|\mathbf{p}|\ell \sim 1$, and then decreases only gradually for larger $|\mathbf{p}|$. Such profiles of $\epsilon(\mathbf{p}, 0)$ and $\alpha_e[\mathbf{p}]$ in Fig. 2 are qualitatively quite similar to those of monolayer graphene studied earlier.²³

Still there are some clear differences: (1) Note that the basic Landau gap of bilayer graphene, $\omega_c \approx 45 B[\text{T}]$ K, is about one-order of magnitude smaller than the monolayer gap $\omega_c^{\text{mono}} \approx 400 \sqrt{B[\text{T}]} \text{ K}$ at $B=1\text{T}$, or

$$\omega_c/\omega_c^{\text{mono}} \approx 0.1 \sqrt{B[\text{T}]}. \quad (3.15)$$

Numerically it turns out that, for both monolayer and bilayer graphene, the vacuum susceptibility α_e^{vac} is around 3 in units of $e^2/(2\pi\omega_c)$ and the peak value of $\epsilon(\mathbf{p}, 0)|_{\nu=0} - 1$ is around 1.8 in units of $\sqrt{2}\alpha/(\epsilon_b\omega_c\ell)$. This actually means that $\epsilon(\mathbf{p}, 0)$ and $\alpha_e[\mathbf{p}]$ are numerically more sizable for bilayers than for monolayers. In particular, for the bilayer the peak value of $\epsilon(\mathbf{p}, 0)$ in Fig. 2 (c) would range from $\epsilon(\mathbf{p}, 0) \approx 9.5 \sim 18$ for $\nu = 0 \sim 12$, with the choice $\epsilon_b \approx 4$ and $B = 1\text{T}$, or $\sqrt{2}\alpha/(\epsilon_b\omega_c\ell) \approx 5.1$. (The value of the substrate dielectric constant ϵ_b depends on the structure of the sample; with SiO_2 on both sides of the bilayer, e.g., one can set $\epsilon_b \approx \epsilon_{\text{SiO}_2} \approx 4$.) This implies that the Coulomb interaction is very efficiently screened (and weakened) in bilayer graphene.

(2) For graphene, unlike standard QH systems, even the $\nu = 0$ vacuum state has an appreciable amount of polarization $\epsilon(\mathbf{p}, 0) - 1$ over a wide range of wavelengths, which reflects the quantum fluctuations or "echoes" of

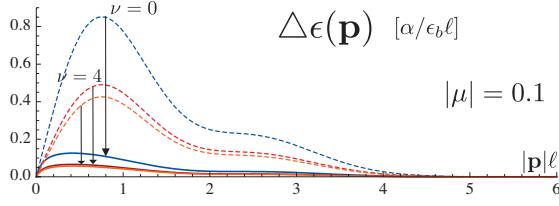


FIG. 3: (Color online) Exciton spectra at $\nu=0$ and 4 for $|\mu| = 0.1$, in units of $\alpha/\epsilon_b\ell$. The spectra are reduced from the bare ones (dashed curves) via screening. At $\nu = 4$ there are two spectra with excitation gaps $\sim \omega_c(\sqrt{2+\mu^2} \mp |\mu|)$.

the Dirac sea in response to an applied field. The echoes are in a sense "harder" for monolayer graphene²³ for which $\epsilon(\mathbf{p}, 0)$ appears almost constant over the wave-vector range in Fig. 2 (c). The rise of the peak values of $\epsilon(\mathbf{p}, 0) - 1$ with filling factor ν is more prominent for monolayers than bilayers; see Eq. (4.6). These features reflect the difference in the underlying Landau-level structures.

(3) In bilayer graphene, unlike in monolayers, the effects of the zero-mode Landau levels become visible as the band gap $\propto 2|\mu|$ develops and this makes the $\nu = 0$ and $\nu = 4$ states distinguishable in $\epsilon(\mathbf{p}, 0)$ and $\alpha_e[\mathbf{p}]$.

In the RPA the response function is written as $P_{\text{RPA}}(\mathbf{p}, \omega) = P(\mathbf{p}, \omega)/\epsilon(\mathbf{p}, \omega)$, from which one can derive²⁸ the inter-Landau-level excitation spectra corrected by the Coulomb interaction. Isolating from $P(\mathbf{p}, \omega)$ one of its poles at $\omega \sim \epsilon_k - \epsilon_n$ and setting $\epsilon(\mathbf{p}, \omega) = 1 - v_{\mathbf{p}} P(\mathbf{p}, \omega) \rightarrow 0$ fixes the pole position of $P_{\text{RPA}}(\mathbf{p}, \omega)$, $\epsilon_{k,n}^{\text{RPA}} = \epsilon_k - \epsilon_n + \Delta\epsilon_{k,n}(\mathbf{p})$ with

$$\Delta\epsilon_{k,n}(\mathbf{p}) \approx \frac{\alpha}{\epsilon_b\ell} \frac{\ell|\mathbf{p}|}{2\epsilon(\mathbf{p}, 0)} \nu_g(\mathbf{p}),$$

$$\nu_g(\mathbf{p}) = \sum g_{nk}(-\mathbf{p})g_{kn}(\mathbf{p}) e^{-x}/x, \quad (3.16)$$

where $x = \ell^2\mathbf{p}^2/2$.

The vacuum state at $\nu = 0$ supports excitons associated with the $n = -2 \rightarrow (1, 0_+)$ or $(-1, 0_-) \rightarrow 2$ transitions with the excitation gap $\epsilon \approx \omega_c(\sqrt{2+\mu^2} + \mu)$ and

$$\nu_g(\mathbf{p}) = 4(1 - \mu/\sqrt{2+\mu^2})h(x), \quad (3.17)$$

where $h(x) = e^{-x}\{1 + \frac{1}{4}x(x-3)\}$. The $\nu = 4$ state supports excitons with energy $\epsilon \approx \omega_c(\sqrt{2+\mu^2} \pm \mu)$ and $\nu_g(\mathbf{p}) = 2(1 \mp \mu/\sqrt{2+\mu^2})h(x)$, respectively; see Fig. 1. Similarly, at $\nu = 4n$ ($n \geq 2$) the $n \rightarrow n+1$ transition with a gap $\Delta\omega_c^{(n)} \approx \epsilon_{n+1} - \epsilon_n$ gives rise to excitons with $\nu_g(0) = (\sqrt{n-1} + \sqrt{n+1})^2$. As a result of screening, $\Delta\epsilon_{k,n}(\mathbf{p}) \propto 1/\epsilon(\mathbf{p}, 0)$, one would observe prominent reduction in magnitude of the exciton spectra; see Fig. 3.

IV. EFFECTIVE GAUGE THEORY

In this section we study low-energy response of bilayer graphene and construct an effective gauge theory. Let us consider the Hall conductance which is read from a response of the form $\frac{1}{2}\sigma_H(A_x\partial_t A_y - A_y\partial_t A_x)$; it is calculated from the current-current correlation function and from Berry's phase as well.¹⁴ A direct calculation similar to the one in the monolayer case²³ shows that $\sigma_H \rightarrow e^2\ell^2$ per electron so that $\sigma_H \rightarrow e^2/(2\pi\hbar) = e^2/h$ per filled level. The result is independent of λ and z .

Care is needed to determine σ_H carried by the vacuum state. As before, we truncate the spectrum and find that the bottom of the Dirac sea contributes $\sim -(N-1)e^2/h$. This yields the vacuum Hall conductance

$$\sigma_H^{\text{vac}}(\xi = \pm 1) = \mp e^2/h \quad (4.1)$$

per valley and spin. This implies that nonzero current and charge are induced^{12,13,14} in the vacua at the two valleys, but they combine to vanish in the vacuum, leaving no observable effect. In general, the Hall conductance σ_H is cast in the form of a spectral asymmetry and the vacuum Hall conductance σ_H^{vac} is related to half of the index (2.7).

The long-wavelength response of bilayer graphene with uniform electron density $\bar{\rho}$ is now summarized by the Lagrangian

$$L_A = \bar{\rho}eA_0 - e^2\ell^2\bar{\rho}\frac{1}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$$

$$+ \frac{1}{2}\alpha_e\mathbf{E}_\parallel^2 - \frac{1}{2}\alpha_m(A_{12})^2. \quad (4.2)$$

Here A_0 detects the charge density $\bar{\rho}$, α_e is the electric susceptibility in Eq. (3.12) and $\alpha_m \propto e^2\ell^2\bar{\rho}/m^*$ stands for the magnetic susceptibility probed by a local variation $A_{12} = \partial_x A_y - \partial_y A_x$ about B .

This response is essentially the same as that for monolayer graphene,²³ except for the values of α_e and α_m . Accordingly the effective theory also takes the same form, i.e., a theory of a vector field $b_\mu = (b_0, b_1, b_2)$, with the Lagrangian to $O(\partial^2)$,

$$L_{\text{eff}}[b] = -eA_\mu\epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda + \frac{1}{\ell^2}b_0 + \frac{1}{2\ell^2\bar{\rho}}b_\mu\epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda$$

$$+ \frac{1}{2\ell^2\bar{\rho}\omega_{\text{eff}}}(b_{k0})^2 - \frac{1}{2}\delta b_{12}v\delta b_{12} \quad (4.3)$$

and the effective cyclotron frequency

$$\omega_{\text{eff}} = e^2\ell^2\bar{\rho}/\alpha_e = \omega_c g(\nu),$$

$$g(\nu) = \frac{\nu}{4F(\mu^2) + \nu_{\{1,0\}}|\mu| + \sum_{n \geq 2} \nu_n \beta_n}, \quad (4.4)$$

where $b_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$; $\ell^2\bar{\rho} = \nu/(2\pi)$ and $\nu = \sum \nu_n$. An advantage of bosonization^{29,30} is that it allows one to handle the Coulomb interaction exactly; it is included in Eq. (4.3) with shorthand notation $\delta b_{12}v\delta b_{12} =$

$\int d^2\mathbf{y} \delta b_{12}(x) v(\mathbf{x}-\mathbf{y}) \delta b_{12}(y)$ and $\delta b_{12} = b_{12} - \bar{\rho}$. (We have omitted the α_m term from $L_{\text{eff}}[b]$ since the Coulomb interaction overtakes it at long wavelengths.)

This effective Lagrangian not only reproduces the original response (4.2), but also shows that the Coulomb interaction $v_{\mathbf{p}} = 2\pi\alpha/(\epsilon_b|\mathbf{p}|)$ substantially modifies the dispersion of the cyclotron mode at long wavelengths $\mathbf{p} \rightarrow 0$,

$$\omega(\mathbf{p}) \approx \omega_{\text{eff}} + \frac{1}{2}(\ell^2 \bar{\rho} v_{\mathbf{p}} + \dots) \mathbf{p}^2, \quad (4.5)$$

where \dots involves $\alpha_m/(\alpha_e \omega_{\text{eff}})$, if α_m is recovered. This excitation spectrum is in good agreement with the RPA result (3.16) at long wavelengths; the filling factor $\nu = 2\pi\ell^2 \bar{\rho}$ in Eq. (4.5) corresponds to $\nu_g(0)$ in Eq. (3.16).

For graphene the Landau levels are not equally spaced and the excitation gaps depend on the level index n or ν . At $\nu = 4n$ ($n \geq 1$), e.g., the minimum gap is $\Delta\omega_c^{(n)} = (\sqrt{n(n+1)+\mu^2} - \sqrt{n(n-1)+\mu^2})\omega_c$, and ω_{eff} in Eq. (4.4) represents such an activation gap.

In particular, at $\nu = 4$ where the zero-mode levels are filled, the effective gap $\propto g(4) = 1/[F(\mu) + \mu]$ derives solely from the vacuum fluctuations; it deviates from the true gap $\Delta\omega_c^{(1)} = (\sqrt{2+\mu^2} - \mu)\omega_c$ by about 20% for $\mu = 0$ and by 5% or less for $\mu > 0.4$. At $\nu = 8$, ω_{eff} and $\Delta\omega_c^{(2)}$ agree within 1.1% for all μ . The agreement is almost exact for higher ν . It is somewhat surprising that, as in the monolayer case, an effective theory constructed from the long-wavelength response alone gives an excellent description of the excitation spectrum. Here we verify again from Eq. (4.4) that the susceptibility α_e of a QH system is generally given by a ratio of the Hall conductance to the Landau gap,²³

$$\alpha_e = \nu(e^2/h)/\omega_{\text{eff}} \approx \sigma_H/\Delta\omega_c^{(n)}. \quad (4.6)$$

This implies that α_e depends on ν and μ nontrivially while it grows like $\alpha_e \propto \nu^{3/2}$ for monolayer graphene.

V. ZERO-MODE LANDAU LEVELS

In this section we take a close look into the properties of the zero-mode levels at each valley, with a tiny splitting $\omega_c \mu z \ll \omega_c$. Special care is needed in considering the response from such an almost degenerate sector, since even a weak field applied as a probe may affect the true eigenmodes. The first task therefore is to resolve the degeneracy of the $(0'_+, 1')$ sector by diagonalizing the external probe plus the $O(\mu z)$ asymmetry,

$$\delta H = - \int d^2\mathbf{x} e A_0 \rho + \int dy_0 \psi^\dagger \Delta \mathcal{H}_{\text{as}} \psi, \quad (5.1)$$

where $\Delta \mathcal{H}_{\text{as}} = \omega_c \mu z \text{diag}[-a^\dagger a, a a^\dagger]$. We suppose that A_0 is slowly varying in space and retain only terms up to first order in $\partial_{\mathbf{x}}$, i.e., $\mathbf{E} = -\partial_{\mathbf{x}} A_0$. In practice, it is convenient, without loss of generality, to take $A_0(\mathbf{x}, t) \rightarrow A_0(y, t)$, or $\mathbf{E} \parallel E_y$.

We take the linear kinetic term $\propto \lambda$ into account (since it may potentially be important at such a low energy scale). Within the $(0'_+, 1')$ sector δH effectively turns³¹ into the matrix Hamiltonian (in y_0 space)

$$-e A_0(y_0, t) - \kappa b_+(\lambda) + \begin{pmatrix} \kappa b_-(\lambda) & \mathcal{E} c(\lambda) \\ \mathcal{E} c(\lambda) & -\kappa b_-(\lambda) \end{pmatrix}, \quad (5.2)$$

with $\mathcal{E} \equiv e\ell E_y/\sqrt{2}$, $\kappa = (1/2)\omega_c \mu z$; $c(\lambda) = \sum_{n=0}^{\infty} \sqrt{3n+1} \alpha_n \zeta_n / (N_0 N_1)$. Here $b_{\pm}(\lambda) = b_1(\lambda) \pm b_0(\lambda)$ in terms of b_0 and b_1 defined in Eq. (2.9). Numerically,

$$\begin{aligned} c(\lambda) &= 1 - 3 \times 10^{-5} \lambda^4 + \dots, \\ b_+(\lambda) &= 1 + \lambda^2 + 0.086 \lambda^4 + \dots, \\ b_-(\lambda) &= 1 - 0.014 \lambda^4 + \dots. \end{aligned} \quad (5.3)$$

The $O(\lambda^4)$ corrections are practically negligible for $\lambda \sim 0.3$. We therefore set $c(\lambda) \approx b_-(\lambda) \approx 1$ and may keep the $O(\lambda^2)$ correction to $b_+(\lambda)$.

The Hamiltonian (5.2) leads to the level splitting $\pm\sqrt{\kappa^2 + \mathcal{E}^2}$ with the new eigenmodes $0''_+$ and $1''$ related to $0'_+$ and $1'$ via the unitary transformation $\psi_{0''_+} = \psi_{0'_+} \cos \theta - \psi_{1'} \sin \theta$, with $\tan 2\theta = e\ell E_y/(\sqrt{2}\kappa)$. Here we see that an inplane electric field enhances the splitting of the zero-mode levels, with a gap

$$\Delta\epsilon = \omega_c \sqrt{(\mu z)^2 + 2e^2 \ell^2 \mathbf{E}_{\parallel}^2 / \omega_c^2}. \quad (5.4)$$

This shows, unexpectedly, that the zero-mode level gap is controllable with an injected Hall current.

For a rough estimate let us note the following: If we take $\mu \sim 0.05$, the level gap is $\omega_c \mu z \sim 5 \times 10^{-3} \text{meV}$ at $B = 1\text{T}$, with $\mu z \sim 1.3 \times 10^{-3} B[\text{T}]$; an applied field of strength $E_{\parallel} \sim 1\text{V/cm}$ leads to a level gap of roughly the same magnitude, as seen from

$$\sqrt{2}e\ell|\mathbf{E}|/\omega_c \sim 10^{-3} \times E[\text{V/cm}]/B[\text{T}]^{3/2}. \quad (5.5)$$

One can calculate the spectral weight $\sim |\langle 0'' | \rho_{\mathbf{p}} | 1'' \rangle|^2$ and derive the susceptibility function associated with the $1'' \rightarrow 0''_+$ transition,

$$(\alpha_e[\mathbf{p}])_{1''}^{0''_+} = \frac{g_s e^2}{2\pi \Delta\epsilon} e^{-x} \frac{1 + (x/4) \mathcal{E}^2 / \kappa^2}{1 + \mathcal{E}^2 / \kappa^2}, \quad (5.6)$$

with $x = \ell^2 \mathbf{p}^2 / 2$ and spin degeneracy $g_s = 2$. (Here we have set $\lambda \rightarrow 0$ for clarity; there is no appreciable change for $\lambda \sim 0.3$.) This $\alpha_e[\mathbf{p}]$, with its scale set by the tiny gap $\Delta\epsilon \ll \omega_c$, essentially governs the dielectric property of the $\nu = 2$ state.

Figure 4 shows $\alpha_e[\mathbf{p}]$ and the associated dielectric function $\epsilon(\mathbf{p}, 0) - 1$ at $\nu = \nu_{1''} = 2$. It is seen that both of them decrease rapidly as \mathbf{E}_{\parallel} becomes strong in the sequence $\sqrt{2}e\ell|\mathbf{E}_{\parallel}|/(\omega_c|\mu|z) = 0, 1, 2, 5$; this is because the $1''$ and $0''_+$ modes are chosen so as to diagonalize $\int d^2\mathbf{x} A_0 \rho$ (for $\kappa \rightarrow 0$). In Fig. 4 both $\alpha_e[\mathbf{p}]$ and $\epsilon(\mathbf{p}, 0) - 1$ are plotted in units $1/|\mu z| \sim (40/|\mu|)(1/B[\text{T}])$ times as large as those in Fig. 2. Numerically, for $\mu \sim 0.05$ the

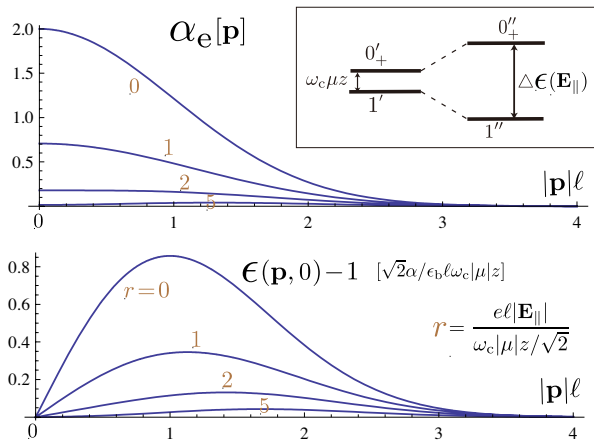


FIG. 4: (Color online) $\alpha_e[\mathbf{p}]$ and $\epsilon(\mathbf{p}, 0) - 1$ at $\nu = 2$, in units of $e^2/(2\pi\omega_c|\mu|z)$ and $\sqrt{2}\alpha/(\epsilon_b\ell\omega_c|\mu|z)$, respectively, for inplane electric fields of strength $\sqrt{2}e\ell|\mathbf{E}_{\parallel}|/(\omega_c|\mu|z) = 0, 1, 2$ and 5. An inset depicts an enhancement of level splitting by an inplane field.

level gap ~ 60 mK at $B = 1$ T would become as large as ~ 300 mK (~ 6.2 GHz h) by an inplane field of $E_{\parallel} \sim 5$ V/cm. This would lead to $\alpha_e[0] \sim 14$ and (the peak value of $\epsilon(\mathbf{p}, 0) - 1 \sim 30$ in common units, i.e., one-order of magnitude larger than those in Fig. 2. This suggests that the dielectric effect would show a marked enhancement around $\nu \sim 2$ if the level gap is resolved by an inplane field at very low temperatures.

Actually, a larger gap, which leads to better level resolution, tends to suppress the effect. Alternatively it will be more practical to observe the field-induced level gap $\Delta\epsilon(\mathbf{E}_{\parallel})$ via the QH effect with an injected current; one would be able to resolve the $\nu = \pm 2$ Hall plateaus using a suitably strong current (that may even suppress the dielectric effect). It will also be possible to detect the level gap directly by micro-wave absorption or via conductance modulations³² associated with it.

VI. SUMMARY AND DISCUSSION

In this paper we have studied the electromagnetic response of bilayer graphene in a magnetic field at integer filling factor ν , with emphasis on clarifying the similarities and differences in quantum features between monolayers and bilayers. Bilayer graphene has a unique feature that its band gap is externally controllable; this makes bilayers richer in electronic properties.

The particle-hole picture of the vacuum state is one of the basic features specific to graphene, both monolayers and multilayers, and is not shared with conventional QH

systems. In graphene even the vacuum state responds to an external field and acts as a dielectric medium, with the Coulomb interaction being efficiently screened over the scale of the magnetic length. Graphene bilayers and monolayers are qualitatively quite similar in the vacuum dielectric characteristics (apart from some differences that reflect the underlying Landau-level structures), as we have seen in Sec. III, but numerically the dielectric effect (for the vacuum and for $\nu \neq 0$ as well) is generally much more sizable for bilayers because of the difference in the basic Landau gap, $\omega_c^{\text{bi}}/\omega_c^{\text{mono}} \approx 0.1\sqrt{B[\text{T}]}$.

The presence of the zero-energy Landau levels is another basic feature specific to graphene. The monolayer supports four zero-mode levels (one per valley and spin), while the bilayer supports eight such levels (two per valley and spin). The zero-mode levels carry normal Hall conductance e^2/h per level, but in monolayers their effect is hardly visible in density response. In contrast, in bilayer graphene a gate-controlled interlayer field acts to open a band gap between the zero-mode levels at the two valleys and this valley gap, in a sense, activates them: The dielectric effect due to the pseudo-zero-mode levels grows linearly with the band gap and becomes dominant for large gaps, as we have seen in Sec. III.

A finite band gap introduces an asymmetry in the zero-mode spectrum while it leaves other levels practically symmetric. The two zero-mode levels (per spin) at each valley thereby remain degenerate, apart from a tiny kinetic splitting. This robustness in the zero-mode degeneracy at each valley as well as the presence of the zero modes itself is a consequence of the nonzero index (2.7) of the basic bilayer Hamiltonian (with $A_0 \rightarrow 0$ and $z \rightarrow 0$). In Sec. V we have pointed out that this tiny zero-mode level gap at each valley is enhanced (and even controlled) by an inplane electric field or by an injected current. Such a gap, if properly enhanced by an injected current, may be detected via the QH effect or directly via micro-wave absorption.

Finally, in Sec. IV we have noted that the low-energy characteristics of bilayer graphene are neatly summarized by an effective Chern-Simons gauge theory, which accommodates graphene and standard QH systems equally well.

Acknowledgments

The authors wish to thank A. Sawada for useful discussions, especially on detection of field-induced level splitting. This work was supported in part by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture of Japan (Grant No. 17540253).

¹ K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and

A. A. Firsov, Nature (London) **438**, 197 (2005).

² Y. Zhang, Y.-W. Tan, H. L. Stormer, and P. Kim, Nature

- (London) **438**, 201 (2005).
- ³ Y. Zhang, Z. Jiang, J.P. Small, M.S. Purewal, Y.-W. Tan, M. Fazlollahi, J.D. Chudow, J.A. Jaszczak, H. L. Stormer, and P. Kim, Phys. Rev. Lett. **96**, 136806 (2006).
 - ⁴ Y. Zheng and T. Ando, Phys. Rev. B **65**, 245420 (2002).
 - ⁵ V. P. Gusynin and S. G. Sharapov, Phys. Rev. Lett. **95**, 146801 (2005).
 - ⁶ N. M. R. Peres, F. Guinea, and A. H. Castro Neto, Phys. Rev. B **73**, 125411 (2006).
 - ⁷ K. Nomura and A. H. MacDonald, Phys. Rev. Lett. **96**, 256602 (2006).
 - ⁸ J. Alicea and M. P. A. Fisher, Phys. Rev. B **74**, 075422 (2006).
 - ⁹ Note in this connection a proposal to simulate the Klein paradox or tunneling in graphene: M. I. Katsnelson, K. S. Novoselov, and A. K. Geim, Nat. Phys. **2**, 625 (2006).
 - ¹⁰ R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976); A. N. Redlich, Phys. Rev. Lett. **52**, 18 (1984); R. Jackiw, Phys. Rev. D **29**, 2375 (1984).
 - ¹¹ A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983).
 - ¹² G. W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984).
 - ¹³ F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
 - ¹⁴ N. Fumita and K. Shizuya, Phys. Rev. D **49**, 4277 (1994).
 - ¹⁵ E. McCann and V. I. Fal'ko, Phys. Rev. Lett. **96**, 086805 (2006).
 - ¹⁶ M. Koshino and T. Ando, Phys. Rev. B **73**, 245403 (2006).
 - ¹⁷ T. Ohta, A. Bostwick, T. Seyller, K. Horn, and E. Rotenberg, Science **313**, 951 (2006).
 - ¹⁸ E. McCann, Phys. Rev. B **74**, 161403(R) (2006).
 - ¹⁹ E. V. Castro, K. S. Novoselov, S. V. Morozov, N.M.R. Peres, J.M.B. Lopes dos Santos, J. Nilsson, F. Guinea, A. K. Geim, and A. H. Castro Neto, Phys. Rev. Lett. **99**, 216802 (2007).
 - ²⁰ H. Min, B. Sahu, S. K. Banerjee, and A. H. MacDonald, Phys. Rev. B **75**, 155115 (2007).
 - ²¹ J. B. Oostinga, H. B. Heersche, X. Liu, A. F. Morpurgo, and L. M. K. Vandersypen, Nature Materials **7**, 151 (2007).
 - ²² T. Ando, J. Phys. Soc. Jpn. **75**, 074716 (2006); E. H. Hwang and S. Das Sarma, Phys. Rev. B **75**, 205418 (2007); B. Wunsch, T. Stauber, F. Sols, and F. Guinea, New. J. Phys. **8**, 318 (2006).
 - ²³ K. Shizuya, Phys. Rev. B **75**, 245417 (2007); Phys. Rev. B **77**, 075419 (2008).
 - ²⁴ L. M. Malard, J. Nilsson, D. C. Elias, J. C. Brant, F. Plentz, E. S. Alves, A. H. Castro Neto, M. A. Pimenta, Phys. Rev. B **76**, 201401(R) (2007).
 - ²⁵ S. M. Girvin and T. Jach, Phys. Rev. B **29**, 5617 (1984).
 - ²⁶ K. Shizuya, Phys. Rev. B **45**, 11 143 (1992); Phys. Rev. B **52**, 2747 (1995).
 - ²⁷ G. D. Mahan, *Many-Particle Physics*, (Kluwer Academic, Dordrecht/Plenum, New York, 2006).
 - ²⁸ C. Kallin and B. I. Halperin, Phys. Rev. B **30**, 5655 (1984).
 - ²⁹ E. Fradkin and F. A. Schaposnik, Phys. Lett. B **338**, 253 (1995).
 - ³⁰ D.-H. Lee and S.-C. Zhang, Phys. Rev. Lett. **66**, 1220 (1991).
 - ³¹ Note that the matrix element $\langle k, y_0'' | \sum_{\mathbf{p}} (A_0)_{\mathbf{p}\rho-\mathbf{p}} | n, y_0' \rangle$ is rewritten as $e^{-\ell^2 \mathbf{p}^2/4} g_{kn}(\mathbf{p}) \langle y_0'' | A_0(\mathbf{r}, t) | y_0' \rangle$, with $\mathbf{p} = -i\partial/\partial\mathbf{r}$ acting on $A_0(\mathbf{r}, t)$; note also that $g_{00} \approx g_{11} \approx 1$, $g_{1,0+} = e\ell p/\sqrt{2}$ and $g_{0+,1} = e\ell \bar{p}/\sqrt{2}$ to $O(p)$.
 - ³² R. S. Deacon, K.-C. Chuang, R. J. Nicholas, K. S. Novoselov, and A. K. Geim, Phys. Rev. B **76**, 081406(R) (2007).